ACCADEMIA NAZIONALE DEI LINCEI

CONVEGNO

GEOMETRIC THEORY OF PDE'S AND SHARP FUNCTIONAL INEQUALITIES

24 Settembre 2021

A B S T R A C T

Comitato organizzatore: Carlo SBORDONE (Chair), Cristina TROMBETTI (Napoli)

Venerdì 24 Settembre

9.30 Welcome address

9.45 Dorin BUCUR (Université Savoie Mont Blanc, Chambéry): *Multiphase free discontinuity problems*

10.30 Giuseppe BUTTAZZO (Università di Pisa): *Upper and lower bounds for some shape functionals*

11.15 Ilaria FRAGALÀ (Politecnico di Milano): *Rigidity for measurable sets*

12.00 Aldo PRATELLI (Università di Pisa): *Propreties of minimal clusters in the plane with densities*

12.45 Intervallo

14.15 Adriana GARRONI (Sapienza Università di Roma): *Rigidity estimates for incompatible elds and an application to plasticity*

15.00 Carlo NITSCH (Università di Napoli Federico II): *Some optimization. problems in thermal insulation*

15.45 Nicola FUSCO (Linceo, Università di Napoli Federico II): *The isoperimetric inequality outside a convex set: the case of equality*

16.30 Susanna TERRACINI (Università di Torino): *Free boundaries in segregation problems*

17.15 Anna VERDE (Università di Napoli Federico II): *Regularity results for elliptic and parabolic systems with general growth*

POSTER SESSION

18.00 Vincenzo AMATO: *Comparison results for solutions to p-Laplace equations with Robin Boundary conditions*

18.10 Francesca ANGRISANI: *Autonomous and asymptotically quasiconvex functionals with general growth conditions*

18.20 Luca BENATTI: *Minkowski inequality in Riemannian Manifolds with nonnegative Ricci curvature*

18.30 Mattia FOGAGNOLO: *The strictly outward minimizing hull in Riemannian Manifolds and its relation with p-capacities*

18.40 Alba Lia MASIELLO: *Limit, as p tends to infinity, of solutions to p-Laplace Equation with Robin boundary conditions*

18.50 Gloria PAOLI: *A reverse quantitative isoperimetric type inequality for the Dirichlet Laplacian*

19.00 Alberto RONCORONI: *Sharp anisotropic Sobolev inequality in convex cones*

19.10 Rossano SANNIPOLI: *Some properties of the torsion function with Robin boundary conditions*

Il convegno è organizzato in collaborazione con:

Dipartimento di Matematica e Applicazioni 'R. Caccioppoli', Università di Napoli 'Federico II'

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Per accedere al convegno seleziona il seguente link:
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Multiphase free discontinuity problems

Dorin BUCUR (Université Savoie Mont Blanc, Chambéry)

We analyse regularity properties of local solutions to free discontinuity problems characterized by the presence of multiple phases. Among contacts between two different phases, only those which occur at jump points are penalized, leaving for free no-jump interfaces which may occur at the zero level of the corresponding state functions. In this setting, our main result states that the phases are open and the jump set (globally considered for all the phases) is essentially closed and Ahlfors regular. They are consequence of a multiphase monotonicity formula and of a multiphase decay lemma, which extend the corresponding one phase results obtained with Luckhaus and by De Giorgi, Carriero and Leaci, respectively. The proof of the former relies on a sharp collective Sobolev extension result for functions with disjoint supports on a sphere, which may be of independent interest. This is a joint work with I. Fragala and A. Giacomini.

Upper and lower bounds for some shape functionals

Giuseppe BUTTAZZO (Università di Pisa)

The relations between two quantities related to the Laplace operator are considered. In particular, taking as a model the heat diffusion, governed by the heat equation, we aim to study the relations between the average temperature of a heated body and the temperature decay rate of the body in absence of heat sources. The quantities above are expressed by the so-called "torsional rigidity" and by the principal eigenvalue of the Laplace operator. The relations above are studied in the classes of general domains, convex domains, and domains with a small thickness. This allows to obtain a detailed description of the Blasche-Santaló diagram of the two quantities. Several open questions are discussed, in particular when the Laplacian is replaced by the p-Laplacian.

Rigidity for measurable sets

Ilaria FRAGALÀ (Politecnico di Milano)

We discuss the rigidity of measurable subsets in the Euclidean space such that the Lebesgue measure of their intersection with a ball of radius r, centred at any point in the essential boundary, is constant. Based on a joint work with Dorin Bucur.

Propreties of minimal clusters in the plane with densities

Aldo PRATELLI (Università di Pisa)

We consider the isoperimetric problem for clusters in the plane R^2 endowed with two densities, one for the perimeter and one for the area. We can prove that the "usual" property of minimal clusters in the classical setting, that is, the boundaries are given by finitely many C^{1,\alpha} arcs metting in triple points, are still true under natural and very general assumptions. In the isotropic case, that is, if the perimeter density does not depend on the direction of the normal vector, the proof is rather simple, and one can also show that the triple points enjoy the usual 120 degrees Steiner property. The construction for the anisotropic case is much more complicate; moreover, the 120 degrees property is easily seen to fail, but it is still true that the arcs meeting at triple points arrive with three different and well-defined directions. (Based on joint works with V. Franceschi and G. Stefani).

Rigidity estimates for incompatible elds and an application to plasticity

Adriana GARRONI (Sapienza Università di Roma)

Rigidity estimates (as for instance Korn's inequality and its non-linear counterpart) are fundamental tools in many applications involving elasticity. I will discuss the generalisation of these estimates in the context of elastoplasticity and material defects, where the relevant variables are incompatible fields (i.e. vector fields that may be not gradients).

Some optimization. problems in thermal insulation

Carlo NITSCH (Università di Napoli Federico II)

Optimal insulation consists in finding the "best" displacement of a pre- scribed volume of insulating material around a given conductor. According to circumstances, the "best" configuration can be the one which minimizes the heat dispersion, maximizes the heat content, minimizes the heat rate loss etc. We provide a flavor of the state of the art, and then we focus on the case of pre- scribed heat source (inside the conductor), with convective heat transfer across the solid and the environment. This corresponds to consider the stationary heat equation inside both conductor & insulator together with Robin boundary conditions at the external boundary. We aim at maximizing the heat content (the L1 norm of the solution) among all the possible distributions of insulating material with fixed mass, and we prove an optimal upper bound in terms of ge- ometric quantities alone. Eventually we prove a conjecture according to which the ball surrounded by a uniform distribution of insulating material maximizes the heat content.
Joint work with: Francesco della Pietra (Universita` degli Studi di Napoli Federico II), Riccardo Scala (Universita` degli Studi di Siena),
Cristina Trombetti (Universita` degli Studi di Napoli Federico II).

The isoperimetric inequality outside a convex set: the case of equality

Nicola FUSCO (Linceo, Università di Napoli Federico II)

In 2007 Choe, Ghomi and Ritor\'e proved the relative isoperimetric inequality outside a convex set C. Precisely, they showed that if $E\subset{\mathbb R}^N\setminus C$ has finite mass m, then the perimeter of E outside C is greater than or equal to one half the surface measure of the ball with mass $2m$. Moreover, they showed that if C is smooth and the equality holds in this isoperimetric inequality, then E is a half ball sitting on C. In this talk I will discuss the equality case when C is any convex set, not necessarily smooth. This is a joint work with M. Morini.

Free boundaries in segregation problems

Susanna TERRACINI (Università di Torino)

We first consider classes of variational problems for densities that repel each other at distance. Examples are given by the minimizers of Dirichlet functional or the Rayleigh quotient

$$D(\mathbf{u}) = \sum_{i=1}^{k} \int_\Omega |\nabla u_i|^2 \quad \text{or} \quad R(\mathbf{u}) = \sum_{i=1}^{k} \frac{\int_\Omega |\nabla u_i|^2}{\int_\Omega u_i^2}$$

over the class of $H^1(\Omega, \mathbb{R}^k)$ functions attaining some boundary conditions on $\partial\Omega$, and subjected to the constraint

$$\operatorname{dist}(\{u_i > 0\}, \{u_j > 0\}) \geq 1 \qquad \forall i \neq j.$$

As second class of problems, we consider energy minimizers of Dirichlet energies with different metrics

$$D(\mathbf{u}) = \sum_{i=1}^{k} \int_\Omega \langle A_i \nabla u_i, \nabla u_i \rangle$$

with constraint

$$u_i(x) \cdot u_j(x) = 0, \qquad \forall x \in \Omega\,, \forall i \neq j.$$

For these problems, we investigate the optimal regularity of the solutions, prove a free-boundary extremality condition, and derive some preliminary results characterising the emerging free boundary.

References

[1] N. SOAVE, H. TAVARES, S. TERRACINI AND A. ZILIO, *Variational problems with long-range interaction*, Arch. Ration. Mech. Anal. 228 (2018), no. 3, 743-772 (http://arxiv.org/abs/1701.05005)

[2] N. SOAVE AND S. TERRACINI, *An anisotropic monotoncity formula, with applications to some segregation problems*, preprint (2020), (http://arxiv.org/abs/2004.08853)

Regularity results for elliptic and parabolic systems with general growth

Anna VERDE (Università di Napoli Federico II)

In this talk I will present a recent result, obtained in collaboration with T. Isernia and C. Leone, concerning partial Hölder regularity for weak solutions to non-autonomous elliptic systems with general growth. In the second part of my talk I will consider autonomous parabolic systems with general growth proving, also in this case, a partial Hölder regularity result for weak solutions. This second result is obtained in collaboration with M. Foss, T. Isernia and C. Leone.

Poster Session

Comparison results for solutions to p-Laplace equations with Robin Boundary conditions

Vincenzo AMATO

As it's observed in [1], in the last decades comparison results of Talenti type for Elliptic Problems with Dirichlet boundary conditions have been wisely investigated. In this paper, we generalize the results obtained in [1] to the case of p-Laplcace equation with Robin boundary conditions.
The point-wise comparison, obtained in [1] only in the planar case, returns in any dimension when p is sufficiently small.

This is a joint work with Andrea Gentile, Alba Lia Masiello.

References

[1] A.Alvino, C. Nitsch, C. Trombetti- *A Talenti comparison result for solutions to elliptic problems with Robin boundary condition.* Preprint, arXiv (2020).

Autonomous and asymptotically quasiconvex functionals with general growth conditions

Francesca ANGRISANI

We obtain local $C^{1,\alpha}$ regularity for minima u of autonomous integrals in the calculus of variations, assuming ψ-growth hypothesis and $W^{1,\varphi}$-quasiconvexity only asymptotically, both in the sub-quadratic and the super-quadratic case. This result is only obtained at points x_0 such that Du is large enough around x_0. Lipschitz regularity on a dense set follows. The results hold for a large class of Young functions with the Δ_2 condition.

Minkowski inequality in Riemannian Manifolds with nonnegative Ricci curvature

Luca BENATTI

The aim of this talk is to present the sharp Minkowski Inequality proved in a recent paper with M. Fogagnolo and L. Mazzieri. In the setting of complete Riemannian manifold of dimension $n \geq 3$, with nonnegative Ricci curvature and Euclidean Volume Growth we show that the inequality

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma$$

holds true for every bounded open subset Ω with smooth boundary, where H is the mean curvature of the boundary, Ω^* is the striclty outward minismising hull of Ω and $\mathrm{AVR}(g)$ is the Asymptotic Volume Ratio. At the base of the proof there are new Monotonicity Formulas, that, despite the mild regularity of p-harmonic functions, are in force at each level set of the p-capacitary potential associated to Ω. Exploiting this knowledge, we derive the previous inequality using a contradiction argument built on the Iso-p-capacitary inequality, and the convergence of the p-capacity to the perimeter of the strictly outward minimising hull when $p \to 1^+$, as recently shown by M. Fogagnolo and L. Mazzieri.

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The strictly outward minimizing hull in Riemannian Manifolds and its relation with p-capacities

Mattia FOGAGNOLO

This talk discusses some of the results contained in a recent work with L. Mazzieri. We analyze the notion of strictly outward minimizing hull of bounded sets Ω in complete Riemannian manifolds, that can be characterized as the maximal volume solution to the least area problem with obstacle Ω. We find sufficient geometric conditions on the ambient manifold ensuring the well posedness of such optimal envelope, that in general cannot be expected to exist. Moreover, under such conditions, we also show that the perimeter of the strictly outward minimizing hull can be recovered as limit of p-capacities as $p \to 1^+$.

The important class of manifolds with nonnegative Ricci curvature and Euclidean volume growth fulfills the requirements. In fact, in these manifolds, these results give the basis for a recent Minkowski inequality obtained in collaboration with L. Benatti and L. Mazzieri employing the level set flow of the p-capacitary potential of a given bounded domain.

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Limit, as p tends to infinity, of solutions to p-Laplace Equation with Robin boundary conditions

Alba Lia MASIELLO

We study the behavior when $p \to \infty$ of the first p-Laplacian eigenvalue with Robin boundary conditions and of the corresponding eigenfunction. We find that the limit of the eigenfunctions is a viscosity solution to an eigenvalue problem for the so called ∞-laplacian.
Moreover, in the second part of the paper, we focus our attention on the p-Poisson equation for a certain f in $L^\infty(\Omega)$ and we study the limit of solutions when $p \to \infty$.

This is a joint work with Vincenzo Amato, Carlo Nitsch, Cristina Trombetti.

A reverse quantitative isoperimetric type inequality for the Dirichlet Laplacian

Gloria PAOLI

Abstract. A stability result in terms of the perimeter is obtained for the first Dirichlet eigenvalue of the Laplacian operator. In particular we prove that, once we fix the dimension $n \geq 2$, there exists a constant $c > 0$, depending only on n, such that, for every $\Omega \subset \mathbb{R}^n$ convex set with unitary volume, it holds

$$\lambda_1(\Omega) - \lambda_1(B) \geq c\,(P(\Omega) - P(B))^2\,,$$

where by $\lambda_1(\cdot)$ we denote the first Dirichlet eigenvalue of a set, by $P(\cdot)$ its perimeter and by B a ball of $\mathbb{R}^n$ with unitary volume.

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Sharp anisotropic Sobolev inequality in convex cones

Alberto RONCORONI

The starting point of the short talk is the sharp version of the classical Sobolev inequality in $\mathbb{R}^n$ proved in two independent papers: [11] and [1]. The Sobolev inequality has been object of several investigations and generalizations. In particular, in [5] the authors prove a Sobolev-type inequality in $\mathbb{R}^n$ for an anisotropic norm (i.e. a function $H : \mathbb{R}^n \to \mathbb{R}$ convex, positive 1-homogeneous and positive). The proof in [5] is based on the optimal transport technique and leads to the sharp Sobolev inequality. The inspiration for [5] has been the proof, based on the optimal transport, in [9, Appendix] of the isoperimetric inequality (see also [6] for the anisotropic isoperimetric inequality). In [4] we realize that the optimal transport technique can be used to prove a sharp anisotropic Sobolev-type inequality in convex cones of $\mathbb{R}^n$ (see also [8] and [2] for previous results).

An important result related to the Sobolev inequality is the classification of critical points, i.e. entire solutions to the so-called critical p-Laplace equation (see e.g. [7, 3, 10, 12]). If time permits I will present a result related to the anisotropic critical p-Laplace equation in convex cones.

This is based on the paper [4] in collaboration with G. Ciraolo and A. Figalli.

REFERENCES

[1] T. Aubin. *Problèmes isopérimétriques et espaces de Sobolev*. J. Differential Geom. **11** (1976), no. 4, 573–598.

[2] X. Cabré X. Ros-Oton, J. Serra. *Sharp isoperimetric inequalities via the ABP method*. J. Eur. Math. Soc. (JEMS) **18** (2016), no. 12, 2971–2998.

[3] L. Caffarelli, B. Gidas, J. Spruck. *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*. Comm. Pure Appl. Math. **42** (1989), 271–297.

[4] G. Ciraolo, A. Figalli, A. Roncoroni. *Symmetry results for critical anisotropic p-Laplacian equations in convex cones*. Geom. Funct. Anal. **30** (2020), 770–803.

[5] D. Cordero-Erausquin, B. Nazaret, C. Villani. *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*. Adv. Math. **182** (2004), no. 2, 307–332.

[6] A. Figalli, F. Maggi, A. Pratelli. *A mass transportation approach to quantitative isoperimetric inequalities*. Invent. Math. **182** (2010), no. 1, 167–211.

[7] B. Gidas, W. M. Ni, L. Nirenberg. *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$. Mathematical analysis and applications, Part A, 369-402, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981.

[8] P.L. Lions, F. Pacella, M. Tricarico. *Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions*. Indiana Univ. Math. J. **37** (1988), no. 2, 301–324.

[9] V.D. Milman, G. Schechtman. Asymptotic Theory of Finite-Dimensional Normed Spaces, Springer, Berlin, 1986 (with an appendix by M. Gromov).

[10] B. Sciunzi. *Classification of positive* $D^{1,p}(\mathbb{R}^n)$*-solutions to the critical p-Laplace equation in* $\mathbb{R}^n$. Advances in Mathematics **291** (2016), 12–23.

[11] G. Talenti. *Best constant in Sobolev inequality*. Ann. di Mat. Pura ed Appl. **110** (1976), 353–372.

[12] J. Vétois. *A priori estimates and application to the symmetry of solutions for critical p-Laplace equations*. J. Differential Equations **260** (2016), 149–161.

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Some properties of the torsion function with Robin boundary conditions

Rossano SANNIPOLI

In this paper we study some properties of the torsion function with Robin boundary conditions. Here we write the shape derivative of the L^∞ and L^p norms, for $p \geq 1$, of the torsion function, seen as a functional on a bounded simply connected open set $\Omega \subset \mathbb{R}^n$, and prove that the balls are critical shapes for these functionals, when the volume of Ω is preserved.